DEGENERATE P-LAPLACIAN OPERATORS ON H-TYPE GROUPS AND APPLICATIONS TO HARDY TYPE INEQUALITIES

YONGYANG JIN AND GENKAI ZHANG

Abstract. Let $\mathbb{G}$ be a step-two nilpotent group of H-type with Lie algebra $\mathfrak{G} = V \oplus \mathfrak{t}$. We define a class of vector fields $X = \{X_j\}$ on $\mathbb{G}$ depending on a real parameter $k \geq 1$, and we consider the corresponding p-Laplacian operator $L_{p,k}u = \operatorname{div}_X(|\nabla_X u|^{p-2}\nabla_X u)$. For $k = 1$ the vector fields $X = \{X_j\}$ are the left invariant vector fields corresponding to an orthonormal basis of V, for $k = 2$ and $\mathbb{G}$ being the Heisenberg group they are introduced by Greiner [12]. In this paper we obtain the fundamental solution for the operator $L_{p,k}$ and as an application, we get a Hardy type inequality associated with X.

1. Introduction

The study of partial differential operators constructed from non-commutative vector fields satisfying the Hörmander condition [14] has had much development. An important class of such fields, serving as local models, is that of left-invariant vector fields on stratified, nilpotent Lie groups with their associated sub-Laplacians defined by the square-sums of the vector fields. One of the main tools in the study of the regularity theory of the sub-Laplacian equation is the fundamental solution; this has been developed in the works of Folland [5] and [6], Folland and Stein [7], Nagel, Stein and Wainger [17], Rothschild and Stein [19] and Sanchez-Calle [20]. In the papers [2, 13] the authors studied a class of subelliptic p-Laplacians on H-type group associated with the left-invariant vector fields and found the corresponding fundamental solution.

Recently there have been considerable interests in studying the sub-Laplacians as square-sums of vector fields that are not invariant or do not satisfy the Hörmander condition. They turn out to be rather difficult, among the examples of such sub-Laplacians are the Grushin operators and the sub-Laplacian constructed by Kohn [16]. Those non-invariant sub-Laplacians also appear naturally in complex analysis. In the paper [1] Greiner, Beals and Gaveau considered the CR operators $\{Z_j, \bar{Z}_j\}_{j=1}^n$ on $\mathbb{R}^{2n+1}$ as boundary of the complex

1991 *Mathematics Subject Classification.* 35H30, 26D10, 22E25.

Key words and phrases. Fundamental solutions, degenerate Laplacians, Hardy inequality, H-type groups.

Research of Y. Jin supported by the China State Scholarship and G. Zhang by the Swedish Research Council and a STINT Institutional Grant.

domain

$$\left\{(z_1, ...z_{n+1}) \in \mathbb{C}^{n+1}: \quad \mathrm{Im} z_{n+1} > (\sum_{j=1}^{n} |z_j|^2)^k \right\},$$

where $Z_j = \frac{1}{2}(X_j - iY_j)$,

$$X_j = \partial/\partial x_j + 2ky_j|z|^{2k-2}\partial/\partial t, \ Y_j = \partial/\partial y_j - 2kx_j|z|^{2k-2}\partial/\partial t. \tag{1.1}$$

and k is a positive integer. For $k = 1$ these vector fields are left-invariant on the Heisenberg group $\mathbb{R}^{2n+1}$ but for $k \neq 1$ there are neither left nor right-invariant. The fundamental solution for their square sum

$$\sum_{j=1}^{n} Z_j\bar{Z}_j + \bar{Z}_j Z_j$$

is studied in [1]. As is well-known, the explicit formula of the fundamental solution is of substantial importance in the study of boundary $\bar{\partial}$-problem; see e.g. [21]. In [22] Zhang and Niu studied the Greiner vector fields on $\mathbb{R}^{2n+1}$ for general parameter $k \geq 1$ and found the fundamental solution for the degenerate p-subelliptic operators $L_{p,k,}$; see Section 2 below. Note that for non-integral k these vector fields do not satisfy the Hörmander condition and are not smooth.

Heisenberg groups have natural generalizations, namely Carnot groups which are the nilpotent stratified Lie groups $\mathbb{G}$ with Lie algebras $\mathfrak{G} = V_1 \oplus V_2 \oplus \cdots \oplus V_l$ with $[V_i, V_j] \subset V_{i+j}$, with the sub-Riemannian structure defined by the generating subspaces V_1. The sub-Laplacian has generalization to p-sub-Laplacian generated by non-invariant vector fields. The p-sub-Laplacian in this setting plays important role in the study of quasiregular maps [13]. The general theory in this setup is still not fully developed.

An important subclass of Carnot groups is that of H-type groups which were introduced by Kaplan [15] as direct generalizations of Heisenberg groups. In the present paper we will define a class of vector fields X (see (2.3) below) on H-type groups generalizing the vector fields (1.1) considered in [1] and [22], and we find the fundamental solution of the corresponding p-Laplacian with singularity at the identity element. As application we prove a Hardy type inequality associated to X.

Here is a brief review and comparison of our results with those in the literature. The case of Heisenberg groups with general parameter k is done in [22]. When $\mathbb{G}$ is a general Carnot group with the sub-Laplacian being invariant Hardy type inequality has been proved by D'Ambrosio [3]; see also [4] where Hardy type inequalities on Heisenberg groups are studied. Our vector fields are however not invariant and not smooth for non-integral k, and our techniques are slightly different from theirs. In particular the computations in our case

are rather involved; we use some fine structure of the H-type groups and we obtain also the best constant for the Hardy type inequality.

The paper is organized as follows. In Section 2 we recall some basic facts of the H-type group and introduce the degenerate p-Laplacian operator $L_{p,k}$ generalizing the invariant sub-Laplacian; Section 3 is devoted to the proof of the fundamental solution for $L_{p,k}$; In the final Section 4 we prove the Hardy type inequality associated with X.

2. H-type groups and a family of vector fields

We recall that a simply connected nilpotent group $\mathbb{G}$ is of Heisenberg type, or simply H-type, if its Lie algebra $\mathfrak{G} = V \oplus \mathfrak{t}$ is of step-two, $[V, V] \subset \mathfrak{t}$, and if there is an inner product $\langle \cdot\ ,\ \cdot \rangle$ in $\mathfrak{G}$ such that the linear map

$$J : \mathfrak{t} \to \mathrm{End}(V),$$

defined by the relation

$$\langle J_t(u), v\rangle = \langle t, [u, v]\rangle$$

satisfies

$$J_t^2 = -|t|^2 \mathbf{Id}$$

for all $t \in \mathfrak{t}, u, v \in V$. We denote $m = \mathrm{dim} V$ and $q = \mathrm{dim}\mathfrak{t}$.

We identify $\mathbb{G}$ with its Lie algebra $\mathfrak{G}$ via the exponential map, $\exp : V \oplus \mathfrak{t} \to \mathbb{G}$. The Lie group product is given by

$$(u, t)(v, s) = (u + v, t + s + \frac{1}{2}[u, v]). \tag{2.2}$$

Each vector $X \in \mathfrak{G}$ defines a tangent vector at any g by differentiating along $g \cdot \exp(tX)$, namely a left-invariant vector field, denoted also by X. The sub-Laplacian on $\mathbb{G}$ is

$$\Delta_{\mathbb{G}} = \sum_{j=1}^{m} X_j^2,$$

where $\{X_j\}$ is an orthonormal basis of V.

For $g \in \mathbb{G}$, we write $g = (z(g), t(g)) \in V \oplus \mathfrak{t}$, and let $K(g) = (|z(g)|^4 + 16|t(g)|^2)^{\frac{1}{4}}$. In [15] Kaplan proved that there exists a constant $C > 0$ such that the function

$$\Phi(g) = C \cdot K(g)^{2-(m+2q)}$$

is a fundamental solution for the operator $\Delta_{\mathbb{G}}$ with singularity at the identity element. We note that $m + 2q$ is the homogeneous dimension of $\mathbb{G}$.

In [2] the authors considered the following subelliptic p-Laplacian

$$\Delta_p u = \sum_{j=1}^{m} X_j^*(|\nabla_{\mathbb{G}} u|^{p-2} X_j u)$$

on H-type group $\mathbb{G}$, where $\{X_j\}_1^m$ is an orthogonal basis of V, X_j^* is the formal adjoint of X_j, and $\nabla_{\mathbb{G}} = (X_1, ..., X_m)$. For $p = 2$ it is the sub-Laplacian above. They obtained a remarkable explicit formula for the fundamental solution of Δ_p,

$$\Gamma_p = \begin{cases} C_p \ K^{\frac{p-Q}{p-1}} \ , & p \neq Q \\ C_Q \log \frac{1}{K} \ , & p = Q \end{cases}$$

As application, the authors obtained some regularity results for a class of nonlinear subelliptic equations.

Motivated by the work of Greiner, Beals and Gaveau [1], Zhang and Niu [22] considered the following degenerate p-subelliptic operators on the Heisenberg group $\mathbb{R}^{2n+1}$:

$$L_{p,k} u = \mathrm{div}_L(|\nabla_L u|^{p-2} \nabla_L u).$$

Here

$$\nabla_L u = (X_1 u, ..., X_n u, Y_1 u, ..., Y_n u), \quad \mathrm{div}_L(u_1, ..., u_{2n}) = \sum_{j=1}^{n} (X_j u_j + Y_j u_{n+j}),$$

$\{X_j, Y_j\}_{j=1,...,n}$ are the Greiner type vector fields (1.1) for general $k \geq 1$. They obtained a fundamental solution for $L_{p,k}$ at the origin for $1 < p < \infty$,

$$\Gamma_p = \begin{cases} C_{p,k} \ \rho^{\frac{p-Q}{p-1}} \ , & p \neq Q \\ C_{Q,k} \log \frac{1}{\rho} \ , & p = Q \end{cases};$$

where $\rho(z, t) = (|z|^{4k} + t^2)^{1/4k}, Q = 2n + 2k$.

Remark 1. *Note that when $p = 2$ and $k = 1$, $L_{p,k}$ becomes the sub-Laplacian $\Delta_{\mathbb{H}^n}$ on the Heisenberg group $\mathbb{H}^n$. If $p = 2$ and $k = 2, 3, ...$, $L_{p,k}$ is a Greiner operator (see* [1], [12]*). Also we note that vector fields in (1.1) do not possess the translation invariance and they do not satisfy Hörmander's condition for $k > 1, k \notin Z$. Finally we mention that $L_{p,k} u = 0$ is the Euler-Lagrange equation associated to the functional*

$$\int |\nabla_L u|^p, \quad p > 1$$

for functions u satisfying $u, \nabla_L u \in L^p$.

In the present paper we introduce a family of the vector fields $X = \{X_1, ..., X_m\}$ and the corresponding $p-$sub-Laplacian on H-type groups generalizing both of the works above. We let

$$X_j = \partial_j + \frac{1}{2}k|z|^{2k-2}\partial_{[z,e_j]}, \; j = 1, 2..., m, \tag{2.3}$$

where $\partial_j = \partial_{e_j}$, $\partial_{[z,e_j]}$ are the directional derivatives, $\{e_j\}_{j=1,...,m}$ is an orthonormal basis of V and $k \geq 1$ is a fixed parameter. We consider the corresponding degenerate p-Laplacian operator

$$L_{p,k}u = \mathrm{div}_X(|\nabla_X u|^{p-2}\nabla_X u), \tag{2.4}$$

where

$$\nabla_X u = (X_1 u, ..., X_m u), \quad \mathrm{div}_X(u_1, ..., u_m) = \sum_{j=1}^{m} X_j u_j.$$

A natural family of anisotropic dilations attached to $L_{p,k}$ is

$$\delta_\lambda : (z, t) \mapsto (w, s) := (\lambda z, \lambda^{2k} t), \; \lambda > 0, \; (z, t) \in \mathbb{G} = \mathbb{R}^{m+q}. \tag{2.5}$$

It is easy to verify that volume is transformed by δ via

$$dwds = \lambda^Q dzdt,$$

where

$$Q := m + 2kq,$$

which we may call the degree of homogeneity and is the homogeneous dimension in the case $k = 1$. We define a corresponding *homogeneous norm* by

$$d(z, t) := (|z|^{4k} + 16|t|^2)^{1/4k}. \tag{2.6}$$

3. Fundamental Solutions

The main result of this section is the following

Theorem 1. *Let $\mathbb{G}$ be a H-type group identified with its Lie algebra $\mathfrak{G}$ as in (2.2). Then for $1 < p < \infty$,*

$$\Gamma_p = \begin{cases} C_p \; d^{\frac{p-Q}{p-1}} \;, & p \neq Q \\ C_Q \log \frac{1}{d} \;, & p = Q \end{cases}$$

is a fundamental solution of $L_{p,k}$ with singularity at the identity element $0 \in \mathbb{G}$. Here $d(z,t)$ is defined in (2.6),

$$C_p = \frac{p-1}{p-Q}(\sigma_p)^{-\frac{1}{p-1}}, \quad C_Q = -(\sigma_Q)^{-\frac{1}{Q-1}},$$

and

$$\sigma_p = \left(\frac{1}{4}\right)^{q-\frac{1}{2}} \frac{\pi^{\frac{q+m}{2}}\Gamma(\frac{(2k-1)p+m}{4k})}{\Gamma(\frac{m}{2})\Gamma(\frac{(2k-1)p+Q}{4k})}.$$

We prove first some technical identities, which might be of independent interests.

Lemma 1. *Let* $\epsilon > 0$ *and* $d_\varepsilon = (d^{4k} + \varepsilon^{4k})^{\frac{1}{4k}}$. *Then we have*

$$|\nabla_X d_\varepsilon|^2 = \sum_{j=1}^m |X_j(d_\varepsilon)|^2 = \frac{d^{4k}}{d_\varepsilon^{8k-2}}|z|^{4k-2}, \tag{3.7}$$

$$L_{2,k} d_\varepsilon^{4k} = \sum_{j=1}^m X_j^2(d_\varepsilon^{4k}) = 4k(4k-2+Q)|z|^{4k-2}, \tag{3.8}$$

and

$$L_{2,k} d_\varepsilon = \sum_{j=1}^m X_j^2 d_\varepsilon = |\nabla_X d_\varepsilon|^2 \frac{d_\varepsilon^{4k-1}}{d^{4k}} \left\{4k + Q - 2 - (4k-1)d_\varepsilon^{-4k} d^{4k}\right\}. \tag{3.9}$$

Proof. By direct computation,

$$\begin{aligned} X_j(d_\varepsilon) &= \frac{1}{4k} d_\varepsilon^{1-4k} X_j(d_\varepsilon^{4k}) \\ &= \frac{1}{4k} d_\varepsilon^{1-4k}\left[4k|z|^{4k-2}\langle z, e_j\rangle + 16k|z|^{2k-2}\langle t, [z, e_j]\rangle\right] \\ &= d_\varepsilon^{1-4k}\left[|z|^{4k-2}\langle z, e_j\rangle + 4|z|^{2k-2}\langle J_t(z), e_j\rangle\right]. \end{aligned} \tag{3.10}$$

However

$$\langle J_t(z), z\rangle = \langle t, [z, z]\rangle = 0, \quad \langle J_t(z), J_t(z)\rangle = |t|^2|z|^2, \tag{3.11}$$

thus

$$\sum_{j=1}^m \langle z, e_j\rangle\langle J_t(z), e_j\rangle = \langle J_t(z), z\rangle = 0.$$

Consequently

$$\begin{aligned} |\nabla_X d_\epsilon|^2 = \sum_{j=1}^m |X_j(d_\varepsilon)|^2 &= d_\varepsilon^{2-8k}\left[|z|^{8k-4}|z|^2 + 16|z|^{4k-4}|t|^2|z|^2\right] \\ &= \frac{d^{4k}}{d_\varepsilon^{8k-2}}|z|^{4k-2}, \end{aligned} \tag{3.12}$$

proving the first identity. Continuing the previous computation of $X_j d_\epsilon$, we find

$$\begin{aligned}\sum_{j=1}^m X_j^2(d_\varepsilon^{4k}) &= \sum_{j=1}^m X_j\left[X_j(d^{4k})\right]\\ &= \sum_{j=1}^m X_j\left[4k\left(|z|^{4k-2}\langle z,e_j\rangle + 4|z|^{2k-2}\langle J_t(z),e_j\rangle\right)\right]\\ &= 4k\sum_{j=1}^m\left\{(2k-1)|z|^{4k-4}2\langle z,e_j\rangle^2 + |z|^{4k-2}\right.\\ &\qquad \left.+8(k-1)|z|^{2k-4}\langle z,e_j\rangle\langle J_t(z),e_j\rangle + 2k|z|^{4k-4}\langle J_{[z,e_j]}(z),e_j\rangle\right\}.\end{aligned} \tag{3.13}$$

To compute the last term in (3.13), we choose an orthonormal basis $\{t_i\}_{i=1,\dots q}$ of $\mathfrak{t}$, then

$$\begin{aligned}\sum_{j=1}^m\langle J_{[z,e_j]}(z),e_j\rangle &= \sum_{j=1}^m|[z,e_j]|^2 = \sum_{j=1}^m\sum_{i=1}^q\langle t_i,[z,e_j]\rangle^2 = \sum_{i=1}^q\sum_{j=1}^m\langle J_{t_i}(z),e_j\rangle^2\\ &= \sum_{i=1}^q|t_i|^2|z|^2 = q|z|^2.\end{aligned} \tag{3.14}$$

Therefore

$$\begin{aligned}\sum_{j=1}^m X_j^2(d_\varepsilon^{4k}) &= 4k\left\{(4k-2)|z|^{4k-2} + m|z|^{4k-2} + 2k|z|^{4k-4}\cdot q|z|^2\right\}\\ &= 4k(4k-2+Q)|z|^{4k-2},\end{aligned} \tag{3.15}$$

where $Q = m + 2kq$. We can find $X_j^2 d_\epsilon$ in terms of $X_j^2 d_\epsilon^{4k}$ and $|X_j^2 d_\epsilon|^2$. Indeed

$$X_j^2(d_\varepsilon^{4k}) = X_j(4kd_\varepsilon^{4k-1}X_jd_\varepsilon) = 4kd_\varepsilon^{4k-1}X_j^2d_\varepsilon + 4k(k-1)d_\varepsilon^{4k-2}|X_jd_\varepsilon|^2, \tag{3.16}$$

thus

$$\begin{aligned}\sum_{j=1}^m X_j^2 d_\varepsilon &= \frac{1}{4k}d_\varepsilon^{1-4k}\left\{\sum_{j=1}^m X_j^2(d_\varepsilon^{4k}) - 4k(k-1)d_\varepsilon^{4k-2}\sum_{j=1}^m|X_jd_\varepsilon|^2\right\}\\ &= \frac{1}{4k}d_\varepsilon^{1-4k}\left\{4k(4k+Q-2)|z|^{4k-2} - 4k(4k-1)d_\varepsilon^{-4k}d^{4k}|z|^{4k-2}\right\}\\ &= d_\varepsilon^{1-4k}|z|^{4k-2}\left\{4k+Q-2-(4k-1)d_\varepsilon^{-4k}d^{4k}\right\}\\ &= |\nabla_X d_\varepsilon|^2\frac{d_\varepsilon^{4k-1}}{d^{4k}}\left\{4k+Q-2-(4k-1)d_\varepsilon^{-4k}d^{4k}\right\},\end{aligned} \tag{3.17}$$

by using the first identity. $\square$

We prove now Theorem 1.

Proof. We consider the case $1 < p < Q$ first. Denote $d_\varepsilon = (d^{4k} + \varepsilon^{4k})^{\frac{1}{4k}}, \quad \varepsilon > 0$. We compute $L_{p,k}(d_\varepsilon^{\frac{p-Q}{p-1}})$. The function $v = d_\varepsilon^{\frac{p-Q}{p-1}}$ is of the form $v = f \circ d_\varepsilon$ with $f(x) = x^{\frac{p-Q}{p-1}}$. For $f \in C^2(\mathbb{R}^+)$, we have

$$
\begin{aligned}
L_{p,k}(f \circ d_\varepsilon) &= f'|f'|^{p-2}|\nabla_X d_\varepsilon|^{p-2} \sum_{j=1}^m X_j^2 d_\varepsilon + |\nabla_X d_\varepsilon|^{p-2} \sum_{j=1}^m X_j d_\varepsilon \cdot X_j \left(f'|f'|^{p-2}\right) \\
&\qquad + f'|f'|^{p-2} \sum_{j=1}^m X_j d_\varepsilon \cdot X_j \left(|\nabla_X d_\varepsilon|^{p-2}\right) \\
&= I_1 + I_2 + I_3.
\end{aligned} \tag{3.18}
$$

I_1 and I_2 can be found by using the Lemma 1,

$$
\begin{aligned}
I_1 &= f'|f'|^{p-2}|\nabla_X d_\varepsilon|^{p-2}|\nabla_X d_\varepsilon|^2 \frac{d_\varepsilon^{4k-1}}{d^{4k}} \left\{4k + Q - 2 - (4k-1) d_\varepsilon^{-4k} d^{4k}\right\} \\
&= f'|f'|^{p-2}|\nabla_X d_\varepsilon|^p \left\{(4k + Q - 2)\frac{d_\varepsilon^{4k-1}}{d^{4k}} - \frac{(4k-1)}{d_\varepsilon}\right\},
\end{aligned} \tag{3.19}
$$

$$
\begin{aligned}
I_2 &= |\nabla_X d_\varepsilon|^{p-2} \sum_{j=1}^m X_j d_\varepsilon \cdot \left\{f''|f'|^{p-2} X_j d_\varepsilon + (p-2)|f'|^{p-2} f'' X_j d_\varepsilon\right\} \\
&= |\nabla_X d_\varepsilon|^p \left\{f''|f'|^{p-2} + (p-2)|f'|^{p-2} f''\right\} \\
&= |f'|^{p-2}|\nabla_X d_\varepsilon|^p \left\{(p-1) f''\right\}.
\end{aligned} \tag{3.20}
$$

Using $X_j|\nabla_X d_\epsilon|^{p-2} = \frac{p-2}{2}|\nabla_X d_\epsilon|^{p-4} X_j |\nabla_X d_\epsilon|^2$ and the Lemma 1, we find

$$
\begin{aligned}
I_3 &= f'|f'|^{p-2} \sum_{j=1}^m X_j d_\varepsilon \cdot \frac{p-2}{2}|\nabla_X d_\varepsilon|^{p-4} X_j(|\nabla_X d_\varepsilon|^2) \\
&= \frac{p-2}{2} f'|f'|^{p-2}|\nabla_X d_\varepsilon|^{p-4} \sum_{j=1}^m X_j d_\varepsilon \cdot X_j \left(d_\varepsilon^{2-8k} d^{4k} |z|^{4k-2}\right) \\
&= \frac{p-2}{2} f'|f'|^{p-2}|\nabla_X d_\varepsilon|^{p-4} \sum_{j=1}^m X_j d_\varepsilon \cdot \Big\{(2-8k) d_\varepsilon^{1-8k} d^{4k}|z|^{4k-2} X_j d_\varepsilon \\
&\qquad\qquad + 4k d_\varepsilon^{2-8k} d^{4k-1}|z|^{4k-2} X_j d \\
&\qquad\qquad + (4k-2) d_\varepsilon^{2-8k} d^{4k}|z|^{4k-4}\langle z, e_j\rangle\Big\} \\
&= \frac{p-2}{2} f'|f'|^{p-2}|\nabla_X d_\varepsilon|^{p-4} \Big\{(2-8k) d_\varepsilon^{3-16k} d^{8k}|z|^{8k-4} + 4k d_\varepsilon^{3-12k} d^{4k}|z|^{8k-4} \\
&\qquad\qquad + (4k-2) d_\varepsilon^{3-12k} d^{4k}|z|^{8k-4}\Big\} \\
&= (p-2)(4k-1) f'|f'|^{p-2}|\nabla_X d_\varepsilon|^p \frac{\varepsilon^{4k}}{d_\varepsilon\, d^{4k}}.
\end{aligned} \tag{3.21}
$$

Hence,

$$
\begin{aligned}
&L_{p,k}(f\circ d_\varepsilon)\\
&= I_1+I_2+I_3 \qquad (3.22)\\
&= |f'|^{p-2}|\nabla_X d_\varepsilon|^p\left\{(p-1)f''+f'\left[\frac{(Q-1)d^{4k}+(4kp-4k+Q-p)\varepsilon^{4k}}{d_\varepsilon d^{4k}}\right]\right\}.
\end{aligned}
$$

Taking $f(x)=x^{\frac{p-Q}{p-1}}$ $(x>0)$ the above is

$$
\begin{aligned}
L_{p,k}\left(d_\varepsilon^{\frac{p-Q}{p-1}}\right) &= \left|\frac{p-Q}{p-1}d_\varepsilon^{\frac{1-Q}{p-1}}\right|^{p-2}\left(\frac{d^{2k}|z|^{2k-1}}{d_\varepsilon^{4k-1}}\right)^p\left\{\frac{p-Q}{p-1}(1-Q)d_\varepsilon^{\frac{2-p-Q}{p-1}}\right.\\
&\qquad\left.+\frac{p-Q}{p-1}d_\varepsilon^{\frac{1-Q}{p-1}}\left[\frac{(Q-1)d^{4k}+(4kp-4k+Q-p)\varepsilon^{4k}}{d_\varepsilon d^{4k}}\right]\right\}\\
&= -\left(\frac{Q-p}{p-1}\right)^{p-1}d_\varepsilon^{1-Q}\left(\frac{d^{2k}|z|^{2k-1}}{d_\varepsilon^{4k-1}}\right)^p\left\{(4kp-4k+Q-p)\frac{\varepsilon^{4k}}{d^{4k}d_\varepsilon}\right\} \qquad (3.23)\\
&= -\left(\frac{Q-p}{p-1}\right)^{p-1}(4kp-4k+Q-p)\frac{d^{2kp-4k}|z|^{(2k-1)p}\varepsilon^{4k}}{d_\varepsilon^{(4k-1)p+Q}}\\
&= \varepsilon^{-Q}\psi(\delta_{1/\varepsilon}(z,t)),
\end{aligned}
$$

where

$$
\psi(z,t) := -\left(\frac{Q-p}{p-1}\right)^{p-1}(4kp-4k+Q-p)\frac{d^{2kp-4k}|z|^{(2k-1)p}}{(1+d^{4k})^{(4kp-p+Q)/4k}}.
$$

Now for any $\varphi\in C_0^\infty(\mathbb{G})$, it follows that

$$
\begin{aligned}
\langle L_{p,k}(d^{\frac{p-Q}{p-1}}),\varphi\rangle &= \lim_{\varepsilon\to 0}\int_{\mathbb{G}} L_{p,k}(d_\varepsilon^{\frac{p-Q}{p-1}})\varphi\\
&= \lim_{\varepsilon\to 0}\varepsilon^{-Q}\int_{\mathbb{G}}\psi(\delta_{1/\varepsilon}(z,t))\varphi(z,t) \qquad (3.24)\\
&= \lim_{\varepsilon\to 0}\int_{\mathbb{G}}\psi(z,t)\varphi(\varepsilon z,\varepsilon^{2k}t)\\
&= \varphi(0)\int_{\mathbb{G}}\psi(z,t).
\end{aligned}
$$

Finally we evaluate the integral $\int_{\mathbb{G}}\psi(z,t)$. We use the polar coordinates $z=rz^*$ with $r=d$ and $z^*\in S:=\{g\in\mathbb{G}:\ d(g)=1\}$ being the sphere with respect to d . By a general

integral formula on homogeneous groups (see [7], Proposition 1.15) we have

$$
\begin{aligned}
&-\int_{\mathbb{G}} \psi(z,t)\\
&= (4kp-4k+Q-p)\int_{\mathbb{G}} \frac{d^{2kp-4k}|z|^{(2k-1)p}}{(1+d^{4k})^{(4kp-p+Q)/4k}}\\
&= (4kp-4k+Q-p)\int_S |z^*|^{(2k-1)p}\int_0^\infty \frac{r^{-4k-1}}{(1+r^{-4k})^{(4kp-p+Q)/4k}} dr d\sigma\\
&= (4kp-4k+Q-p)\int_S |z^*|^{(2k-1)p} d\sigma \frac{1}{4k}\int_1^\infty t^{\frac{p-Q-4kp}{4k}} dt\\
&= \int_S |z^*|^{(2k-1)p} d\sigma,
\end{aligned}
$$

Denote temporarily $\gamma = (2k-1)p$. We use the usual trick to evaluate the integral on the sphere, replacing it by an integral on the ball,

$$
\begin{aligned}
\int_S |z^*|^\gamma d\sigma &= (Q+\gamma)\int_0^1 r^{\gamma+Q-1} dr \int_S |z^*|^\gamma d\sigma\\
&= (Q+\gamma)\int_S\int_0^1 |rz^*|^\gamma r^{Q-1} dr d\sigma\\
&= (Q+\gamma)\int_{d<1} |z|^\gamma,
\end{aligned}
$$

and furthermore

$$
\begin{aligned}
\int_{d<1} |z|^\gamma &= \int_{|t|<\frac{1}{4}} \int_{|z|<(1-16|t|^2)^{\frac{1}{4k}}} |z|^\gamma dz dt\\
&= \omega_{m-1}\int_{|t|<\frac{1}{4}}\int_0^{(1-16|t|^2)^{\frac{1}{4k}}} r^{\gamma+m-1} dr dt\\
&= \frac{\omega_{m-1}\omega_{q-1}}{\gamma+m}\int_0^{\frac{1}{4}} (1-16s^2)^{\frac{\gamma+m}{4k}} s^{q-1} ds\\
&= \frac{\omega_{m-1}\omega_{q-1}}{2(\gamma+m)}\left(\frac{1}{4}\right)^q \int_0^1 (1-\rho)^{\frac{\gamma+m}{4k}} \rho^{\frac{q-2}{2}} d\rho\\
&= \frac{\omega_{m-1}\omega_{q-1}}{2(\gamma+m)}\left(\frac{1}{4}\right)^q \frac{\Gamma(\frac{\gamma+m+4k}{4k})\cdot\Gamma(\frac{q}{2})}{\Gamma(\frac{\gamma+m+4k+2kq}{4k})}\\
&= \frac{1}{2(\gamma+Q)}\left(\frac{1}{4}\right)^{q-1} \frac{\pi^{\frac{q+m}{2}}\cdot\Gamma(\frac{\gamma+m}{4k})}{\Gamma(\frac{m}{2})\cdot\Gamma(\frac{\gamma+Q}{4k})}.
\end{aligned}
$$

Thus,

$$
\int_S |z^*|^{(2k-1)p} d\sigma = \left(\frac{1}{4}\right)^{q-\frac{1}{2}} \frac{\pi^{\frac{q+m}{2}}\cdot\Gamma(\frac{(2k-1)p+m}{4k})}{\Gamma(\frac{m}{2})\cdot\Gamma(\frac{(2k-1)p+Q}{4k})},
$$

and substituthing this into the previous formula for $-\int_{\mathbb{G}} \psi(z,t)$ we find

$$\int_{\mathbb{G}} \psi(z,t) = -\left(\frac{Q-p}{p-1}\right)^{p-1}\left(\frac{1}{4}\right)^{q-\frac{1}{2}} \frac{\pi^{\frac{q+m}{2}} \cdot \Gamma(\frac{(2k-1)p+m}{4k})}{\Gamma(\frac{m}{2}) \cdot \Gamma(\frac{(2k-1)p+Q}{4k})}$$

proving Theorem 1 for $1 < p < Q$.

An direct examination shows that the formula also holds for $p > Q$, and the critical case $p = Q$ can be treated similarly, we omit the details. □

By a similar method as in Theorem 1, we can also obtain a fundamental solution for a class of weighted p-Laplacian operators on the H-type group $\mathbb{G} = \mathbb{R}^m \oplus \mathbb{R}^q$,

$$L_{p,k,w} = \mathrm{div}_X(|\nabla_X u|^{p-2} w \nabla_X u), \tag{3.25}$$

$$\left(w = d^\alpha |\nabla_X d|^\beta,\ \alpha > -m - 2kq, \beta > \max\left\{\frac{1-Q}{4k-1}, -\frac{m}{2k-1} - 1\right\}\right)$$

where $\{X_j\}_{j=1,\ldots,m}$ is taken from (2.3) and $d(z,t)$ from (2.6).

Theorem 2. *Let $\mathbb{G}$ be the H-type group above $L_{p,k,w}$ the p-sub-Laplacian defined defined as in (3.25). Then for $1 < p < \infty$*

$$\Gamma_{p,w} = \begin{cases} C_{p,w}\ d^{\frac{p-Q-\alpha}{p-1}}, & p \neq Q+\alpha \\ C_{Q+\alpha,w} \log\frac{1}{d}, & p = Q+\alpha \end{cases};$$

is a fundamental solution of $L_{p,k,w}$ with singularity at the identity element $0 \in \mathbb{G}$, where

$$C_{p,w} = \frac{p-1}{p-Q-\alpha}(\sigma_{p,\beta})^{-\frac{1}{p-1}}, \quad C_{Q+\alpha,w} = -(\sigma_{Q+\alpha,\beta})^{-\frac{1}{Q+\alpha-1}},$$

and

$$\sigma_{p,\beta} = \left(\frac{1}{4}\right)^{q-\frac{1}{2}} \frac{\pi^{\frac{q+m}{2}}}{\Gamma(\frac{m}{2})} \frac{\Gamma(\frac{(2k-1)(p+\beta)+m}{4k})}{\Gamma(\frac{(2k-1)(p+\beta)+Q}{4k})}.$$

4. Hardy type inequality

We recall that the classical Hardy inequality states that, for $n \geq 3$,

$$\int_{\mathbb{R}^n} |\nabla\Phi(x)|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_{\mathbb{R}^n} \frac{|\Phi(x)|^2}{|x|^2} dx, \tag{4.26}$$

where $\Phi \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$. It can also be rewritten in terms of certain Schrödinger operator. The inequality and their generalizations are thus of interests in the study of spectral theory of linear and nonlinear partial differential equations (see e. g. [8], [10], [11]).

In [9] Garofalo and Lanconelli established the following Hardy inequality on the Heisenberg group $\mathbb{H} = \mathbb{H}^n$ associated with left-invariant horizontal gradient $\nabla_{\mathbb{H}}$,

$$\int_{\mathbb{H}} |\nabla_{\mathbb{H}}\Phi|^2 dzdt \geq \left(\frac{Q-2}{2}\right)^2 \int_{\mathbb{H}} \left(\frac{|z|^2}{|z|^4+t^2}\right) |\Phi|^2 dzdt, \tag{4.27}$$

where $\Phi \in C_0^\infty(\mathbb{H} \setminus \{0\})$, $Q = 2n+2$ is the homogeneous dimension of $\mathbb{H}$, and $\nabla_{\mathbb{H}}\Phi = (X_1\Phi, X_2\Phi, \cdots, X_n\Phi, Y_1\Phi, \cdots, Y_n\Phi)$, $X_j = \frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t}$, $Y_j = \frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t}$, for $(z,t) \in \mathbb{H}, z = (x,y) \in \mathbb{R}^n \times \mathbb{R}^n$, $t \in \mathbb{R}$. The L^p version of the inequality (4.27) has been obtained, among others, by Niu, Zhang, Wang in [18], which states that for $1 < p < Q$:

$$\int_{\mathbb{H}} |\nabla_{\mathbb{H}}\Phi|^p \geq \left(\frac{Q-p}{p}\right)^p \int_{\mathbb{H}} \left(\frac{|z|}{d}\right)^p \frac{|\Phi|^p}{d^p}. \tag{4.28}$$

In this section we obtain a Hardy type inequality associated with the non-invariant vector fields $X = \{X_j\}$ in (2.3) on the H-type groups by applying the result in Section 3. The inequality in the present paper might be useful in eigenvalue problems and Liouville type theorems for weighted p-Laplacian equation, which we plan to pursue in some subsequent work. Recall the norm d in (2.6).

Theorem 3. *Let $\mathbb{G}$ be the H-type group with the homogeneous dimension $Q = m + 2kq$ and $\alpha \in \mathbb{R}, 1 < p < Q+\alpha$ Then the following inequality holds for $\Phi \in C_0^\infty(\mathbb{G}\backslash\{0\})$,*

$$\int_{\mathbb{G}} d^\alpha |\nabla_X \Phi|^p \geq \left(\frac{Q+\alpha-p}{p}\right)^p \int_{\mathbb{G}} d^\alpha \left(\frac{|z|}{d}\right)^{(2k-1)p} \left|\frac{\Phi}{d}\right|^p. \tag{4.29}$$

Moreover, the constant $(\frac{Q+\alpha-p}{p})^p$ is sharp.

In view of the first equality in Lemma 1 (for $\epsilon = 0$), namely $|\nabla_X d| = (\frac{|z|}{d})^{2k-1}$, the above inequality can also be written as

$$\int_{\mathbb{G}} d^\alpha |\nabla_X \Phi|^p \geq \left(\frac{Q+\alpha-p}{p}\right)^p \int_{\mathbb{G}} d^{\alpha-p} |\nabla_X d|^p |\Phi|^p.$$

Remark 2. *If $q = 1$ and $\alpha = 0$, then our Theorem 3 is actually the Theorem 3.1 in* [22].

For the proof of Theorem 3, we need the following Lemma; see also [18] for the case $w = 1$.

Lemma 2. *Let $w \geq 0$ be a weight function in $\Omega \subset \mathbb{G}$ and $L_{p,k,w}u = div_X(|\nabla_X u|^{p-2} w \nabla_X u)$. Suppose that for some $\lambda > 0$, there exists $v \in C^\infty(\Omega), v > 0$ such that*

$$-L_{p,k,w}v \geq \lambda g v^{p-1} \tag{4.30}$$

for some $g \geq 0$, in the sense of distribution acting on non-negative test functions. Then for any $u \in HW_0^{1,p}(\Omega, w)$, it holds that

$$\int_\Omega |\nabla_X u|^p w \geq \lambda \int_\Omega g|u|^p,$$

where $HW_0^{1,p}(\Omega, w)$ denote the closure of $C_0^\infty(\Omega)$ in the norm $(\int_\Omega |\nabla_X u|^p w)^{\frac{1}{p}}$.

Proof. We take $\frac{\varphi^p}{v^{p-1}}$ as a test function in (4.30), where $\varphi \in C_0^\infty(\Omega), \varphi \geq 0$,

$$I := \int_\Omega w|\nabla_X v|^{p-2}\nabla_X v \cdot \nabla_X \left(\frac{\varphi^p}{v^{p-1}}\right) \geq \lambda \int_\Omega g\varphi^p.$$

We shall prove

$$\int_\Omega w|\nabla_X \varphi|^p - I \geq 0 \tag{4.31}$$

which together with the previous inequality implies Lemma 2 for $u = \varphi \in C_0^\infty(\Omega)$. Now, the above is an integration with integrand (disregarding the common factor w),

$$\begin{aligned} &|\nabla_X \varphi|^p - |\nabla_X v|^{p-2}\nabla_X \left(\frac{\varphi^p}{v^{p-1}}\right) \cdot \nabla_X v \\ &= |\nabla_X \varphi|^p - p\frac{\varphi^{p-1}}{v^{p-1}}|\nabla_X v|^{p-2}\nabla_X \varphi \cdot \nabla_X v + (p-1)\frac{\varphi^p}{v^p}|\nabla_X v|^p \\ &= \frac{1}{v^p}\left(v^p|\nabla_X \varphi|^p + (p-1)\varphi^p|\nabla_X v|^p - pv\varphi^{p-1}|\nabla_X v|^{p-2}\nabla_X \varphi \cdot \nabla_X v\right). \end{aligned} \tag{4.32}$$

We estimate last term from above using the Young's inequality

$$ab \leq \frac{1}{p}\, a^p + (1 - \frac{1}{p})\, b^{\frac{p}{p-1}},$$

and get

$$\begin{aligned} pv\varphi^{p-1}|\nabla_X v|^{p-2}\nabla_X \varphi \cdot \nabla_X v &\leq pv|\nabla_X \varphi| \cdot \varphi^{p-1}|\nabla_X v|^{p-1} \\ &\leq p\ \left[\frac{v^p|\nabla_X \varphi|^p}{p} + \frac{p-1}{p}\varphi^p|\nabla_X v|^p\right] \\ &= v^p|\nabla_X \varphi|^p + (p-1)\varphi^p|\nabla_X v|^p. \end{aligned} \tag{4.33}$$

Hence (4.31) follows. The proof of Lemma 2 is finished by taking $\varphi \to u$. □

We prove now Theorem 3.

Proof. Case (i): $p \neq Q$. We claim that the conditions in Lemma 2 are satisfied with

$$w = d^\alpha, \quad v = d^{\frac{p-Q-\alpha}{p}}, \quad g = d^\alpha \frac{|z|^{(2k-1)p}}{d^{2kp}}, \quad \lambda = \left(\frac{Q+\alpha-p}{p}\right)^p, \quad \Omega = \mathbb{G}\backslash\{0\},$$

which then proves the Theorem. Indeed, for any $\varphi \in C_0^\infty(\mathbb{G}\backslash\{0\})$ we have

$$
\begin{aligned}
\langle -L_{p,k,w}v, \varphi\rangle &= -\left(\frac{Q+\alpha-p}{p}\right)^{p-1}\int_{\mathbb{G}}(d^{\frac{Q+\alpha}{p}-Q}|\nabla_X d|^{p-2}\nabla_X d)\cdot\nabla_X\varphi \\
&= -\left(\frac{Q+\alpha-p}{p}\right)^{p-1}\int_{\mathbb{G}}(d^{1-Q}|\nabla_X d|^{p-2}\nabla_X d)\cdot d^{\frac{Q+\alpha-p}{p}}\nabla_X\varphi \\
&= -\left(\frac{Q+\alpha-p}{p}\right)^{p-1}\int_{\mathbb{G}}(d^{1-Q}|\nabla_X d|^{p-2}\nabla_X d)\cdot\nabla_X(\varphi\cdot d^{\frac{Q+\alpha-p}{p}}) \\
&\quad + \left(\frac{Q+\alpha-p}{p}\right)^{p-1}\int_{\mathbb{G}}(d^{1-Q}|\nabla_X d|^{p-2}\nabla_X d)\cdot\nabla_X(d^{\frac{Q+\alpha-p}{p}})\,\varphi.
\end{aligned}
\tag{4.34}
$$

Denoting $C_{p,Q} = \left|\frac{p-1}{p-Q}\right|^{p-2}\frac{p-1}{p-Q}$ and rewriting

$$
d^{1-Q}|\nabla_X d|^{p-2}\nabla_X d = C_{p,Q}\left|\nabla_X\left(d^{\frac{p-Q}{p-1}}\right)\right|^{p-2}\nabla_X\left(d^{\frac{p-Q}{p-1}}\right)
$$

we see that (4.34) is

$$
\begin{aligned}
&\langle -L_{p,k,w}v, \varphi\rangle \\
&= -C_{p,Q}\left(\frac{Q+\alpha-p}{p}\right)^{p-1}\int_{\mathbb{G}}\left|\nabla_X\left(d^{\frac{p-Q}{p-1}}\right)\right|^{p-2}\nabla_X\left(d^{\frac{p-Q}{p-1}}\right)\cdot\nabla_X\left(\varphi d^{\frac{Q+\alpha-p}{p}}\right) \\
&\quad + \left(\frac{Q+\alpha-p}{p}\right)^{p-1}\int_{\mathbb{G}}\left(d^{1-Q}|\nabla_X d|^{p-2}\nabla_X d\right)\cdot\nabla_X\left(d^{\frac{Q+\alpha-p}{p}}\right)\varphi.
\end{aligned}
\tag{4.35}
$$

However the first integral in (4.35) is zero by Theorem 2, since ϕ is supported away from 0, and we find

$$
\begin{aligned}
\langle -L_{p,k,w}v, \varphi\rangle &= \left(\frac{Q+\alpha-p}{p}\right)^{p-1}\int_{\mathbb{G}} d^{1-Q}|\nabla_X d|^{p-2}\nabla_X d\cdot\nabla_X(d^{\frac{Q+\alpha-p}{p}})\varphi \\
&= \left(\frac{Q+\alpha-p}{p}\right)^{p}\int_{\mathbb{G}} d^{\frac{Q+\alpha}{p}-1-Q}|\nabla_X d|^p\varphi \\
&= \left(\frac{Q+\alpha-p}{p}\right)^{p}\int_{\mathbb{G}} d^\alpha d^{\frac{p-Q-\alpha}{p}(p-1)}\frac{|z|^{(2k-1)p}}{d^{2kp}}\varphi \\
&= \left(\frac{Q+\alpha-p}{p}\right)^{p}\int_{\mathbb{G}} d^\alpha\frac{|z|^{(2k-1)p}}{d^{2kp}}v^{p-1}\varphi,
\end{aligned}
\tag{4.36}
$$

where in the second last equality we have used Lemma 1 that $|\nabla_X d|^p = \left(\frac{|z|}{d}\right)^{(2k-1)p}$. This proves our claim.

Case (ii): $p = Q$.

The proof is almost the same as the above once we notice the following fact: $C_Q \log \frac{1}{d}$ is a fundamental solution of $L_{Q,k}$ on $\mathbb{G}$, and

$$d^{1-Q}|\nabla_X d|^{Q-2}\nabla_X d = -|\nabla_X \log(d^{-1})|^{Q-2}\nabla_X \log(d^{-1}).$$

It remains to show the sharpness of the constant $(\frac{Q+\alpha-p}{p})^p$. This is equivalent to show that any constant $B > 0$ for which the inequality

$$\int_{\mathbb{G}} d^{\alpha}|\nabla_X \Phi|^p \geq B \int_{\mathbb{G}} d^{\alpha-p}|\nabla_X d|^p|\Phi|^p \tag{4.37}$$

holds must satisfy $B \leq (\frac{Q+\alpha-p}{p})^p$. We shall construct a sequence $\{u_j\}_{j=1}^{\infty}$ of functions so that the inequality (4.29) approximates to an identity up to the order $O(1)$ in j. Given any positive integer j it is elementary that there exists ψ_j in $C_0^{\infty}(0,\infty)$ such that $\operatorname{supp}\phi_j = [2^{-j-1}, 2]$, $\psi_j(x) = 1$ on $[2^{-j}, 1]$, and $|\psi_j'(x)| \leq C2^j$ on $[2^{-j-1}, 2^{-j}]$, where C is a constant independent of j. Let

$$u_j(z,t) = d(z,t)^{\frac{p-Q-\alpha}{p} - \frac{1}{j}}\, \psi_j(d(z,t)).$$

Clearly $u_j \in C^{\infty}(G \setminus \{0\})$ and is radial. The gradient is given by

$$\nabla_X u_j = \begin{cases} 0, & 0 \leq d < 2^{-j-1}, \text{ or } d > 2 \\ -(\frac{Q+\alpha-p}{p} + \frac{1}{j}) d^{-\frac{Q+\alpha+p\frac{1}{j}}{p}}\, \nabla_X d, & 2^{-j} < d < 1 \end{cases} \tag{4.38}$$

The left hand side of the above inequality is

$$LHS = \int_{\mathbb{G}} = \int_{2^{-j}<d<1} + \int_{2^{-j-1}<d<2^{-j}} + \int_{1<d<2} = \int_{2^{-j}<d<1} + I + II.$$

The first integration is

$$\int_{2^{-j}<d<1} d^{\alpha}|\nabla_X u_j|^p = (\frac{Q+\alpha-p}{p} + \frac{1}{j})^p \int_{2^{-j}<d<1} d^{-\frac{Q+\alpha+p\frac{1}{j}}{}} |\nabla_X d|^p.$$

This can be computed by using the polar coordinates as in proof of Theorem 1 and is

$$(\frac{Q+\alpha-p}{p} + \frac{1}{j})^p C_0 j,$$

where $C_0 = \frac{(2^p-1)}{p} \int_S |z|^{p(2k+1)}$ (and is evaluated in the proof of Theorem 1). Similarly,

$$RHS = B \int_{2^{-j}<d<1} + III + IV.$$

The first integration is precisely the same as above and is

$$B \int_{2^{-j}<d<1} = B\, C_0 j,$$

with the same constant C_0. It is easy to estimate the error terms and they are all bounded

$$I, II, III, IV \leq C.$$

The inequality (4.37) now becomes

$$(\frac{Q+\alpha-p}{p}+\frac{1}{j})^p C_0 j + I + II \geq BC_0 j + III + IV.$$

Dividing both sides by j and letting $j \to \infty$ prove our claim. $\square$

An immediate consequence of Theorem 3 is the following corollary, known also as the uncertainty principle, this can be proved by estimating the left hand side using Hölder inequality together with inequality (4.29) for $\alpha = 0$.

Corollary 1. *Let $\mathbb{G}$ be the H-type group with the homogeneous dimension $Q = m + 2kq$ associated with the dilations (2.5). $u \in C_0^\infty(\mathbb{G}\backslash\{0\})$, $\frac{1}{s}+\frac{1}{t} = 1$ $(1 < s < Q)$. Then*

$$\left(\int_{\mathbb{G}} |z|^t |u|^t\right)^{\frac{1}{t}} \left(\int_{\mathbb{G}} |\nabla_X u|^s\right)^{\frac{1}{s}} \geq \frac{Q-s}{s} \int_{\mathbb{G}} \frac{|z|^{2k}}{d^{2k}} |u|^2.$$

References

[1] R. Beals, B. Gaveau, and P. Greiner. Uniforms hypoelliptic Green's functions. *J. Math. Pures Appl. (9)*, 77(3):209–248, 1998.

[2] L. Capogna, D. Danielli, and N. Garofalo. Capacitary estimates and the local behavior of solutions of nonlinear subelliptic equations. *Amer. J. Math.*, 118(6):1153–1196, 1996.

[3] Lorenzo D'Ambrosio. Hardy-type inequalities related to degenerate elliptic differential operators. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 4(3):451–486, 2005.

[4] L. D'Ambrozio. Some Hardy inequalities on the Heisenberg group. *Differ. Uravn.*, 40(4):509–521, 575, 2004.

[5] G. B. Folland. A fundamental solution for a subelliptic operator. *Bull. Amer. Math. Soc.*, 79:373–376, 1973.

[6] G. B. Folland. Subelliptic estimates and function spaces on nilpotent Lie groups. *Ark. Mat.*, 13(2):161–207, 1975.

[7] G. B. Folland and E. M. Stein. *Hardy spaces on homogeneous groups*, volume 28 of *Mathematical Notes*. Princeton University Press, Princeton, N.J., 1982.

[8] J. P. García Azorero and I. Peral Alonso. Hardy inequalities and some critical elliptic and parabolic problems. *J. Differential Equations*, 144(2):441–476, 1998.

[9] N. Garofalo and E. Lanconelli. Frequency functions on the Heisenberg group, the uncertainty principle and unique continuation. *Ann. Inst. Fourier (Grenoble)*, 40(2):313–356, 1990.

[10] Jerome A. Goldstein and Ismail Kombe. Nonlinear degenerate parabolic equations on the Heisenberg group. *Int. J. Evol. Equ.*, 1(1):1–22, 2005.

[11] Jerome A. Goldstein and Qi S. Zhang. On a degenerate heat equation with a singular potential. *J. Funct. Anal.*, 186(2):342–359, 2001.

[12] P. C. Greiner. A fundamental solution for a nonelliptic partial differential operator. *Canad. J. Math.*, 31(5):1107–1120, 1979.

[13] J. Heinonen and I. Holopainen. Quasiregular maps on Carnot groups. *J. Geom. Anal.*, 7(1):109–148, 1997.

[14] L. Hörmander. Hypoelliptic second order differential equations. *Acta Math.*, 119:147–171, 1967.

[15] A. Kaplan. Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms. *Trans. Amer. Math. Soc.*, 258(1):147–153, 1980.

[16] J. J. Kohn. Hypoellipticity and loss of derivatives. *Ann. of Math. (2)*, 162(2):943–986, 2005. With an appendix by Makhlouf Derridj and David S. Tartakoff.

[17] A. Nagel, E. M. Stein, and S. Wainger. Balls and metrics defined by vector fields. I. Basic properties. *Acta Math.*, 155(1-2):103–147, 1985.

[18] P. Niu, H. Zhang, and Y. Wang. Hardy type and Rellich type inequalities on the Heisenberg group. *Proc. Amer. Math. Soc.*, 129(12):3623–3630 (electronic), 2001.

[19] L. P. Rothschild and E. M. Stein. Hypoelliptic differential operators and nilpotent groups. *Acta Math.*, 137(3-4):247–320, 1976.

[20] A. Sánchez Calle. Fundamental solutions and geometry of the sum of squares of vector fields. *Invent. Math.*, 78(1):143–160, 1984.

[21] E. M. Stein. Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals. Princeton Mathematical Series, Vol. 43, Princeton University Press, Princeton, New Jersey, 1993,

[22] H. Zhang and P. Niu. Hardy-type inequalities and Pohozaev-type identities for a class of p-degenerate subelliptic operators and applications. *Nonlinear Anal.*, 54(1):165–186, 2003.

Yongyang Jin: Department of Applied Mathematics, Zhejiang University of Technology, Hangzhou, 310032, China; yongyang@zjut.edu.cn.

Genkai Zhang: Mathematical Sciences, Chalmers University of Technology and Mathematical Sciences, Göteborg University, Göteborg, Sweden; genkai@math.chalmers.se